Voting in Two-Crossing Elections

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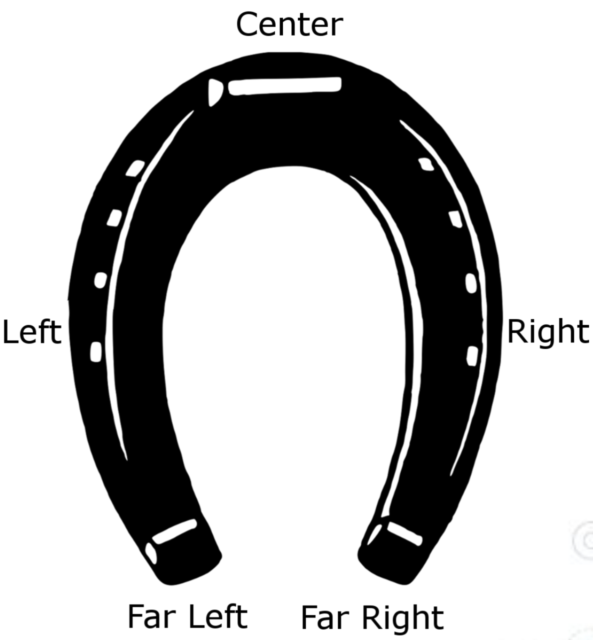
Roger Wattenhofer

Distributed Computing Group
ETH zürich

1.

Motivation

The Horseshoe Theory

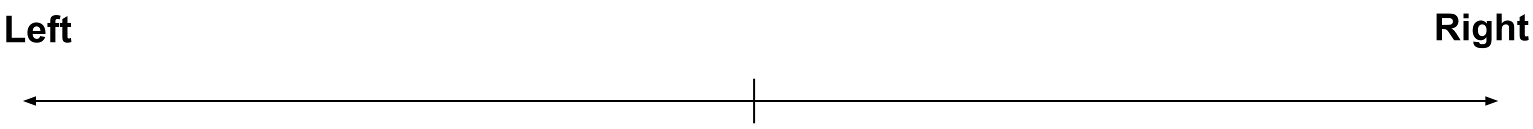


Left-Right Spectrum

Candidate c located at x(c); voter v has ideal point x(v).

Preference by Euclidean distance

Left-Right Spectrum



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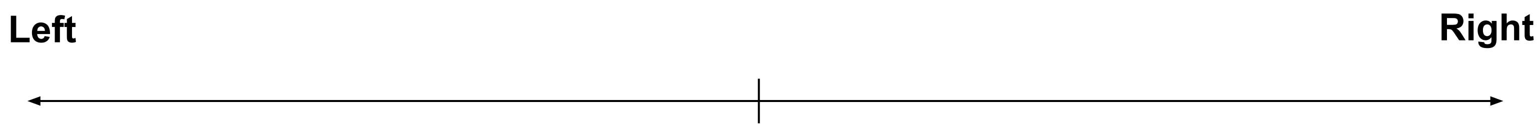

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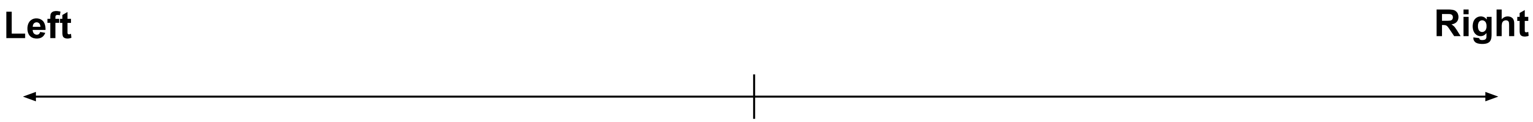

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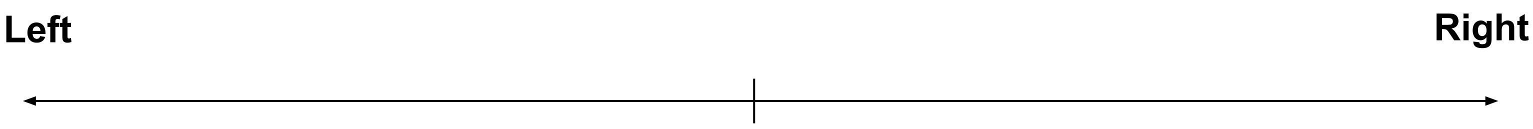

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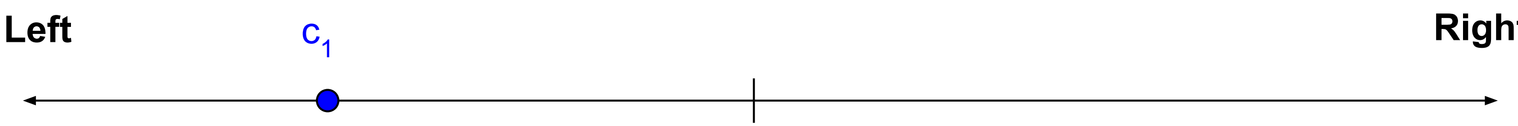

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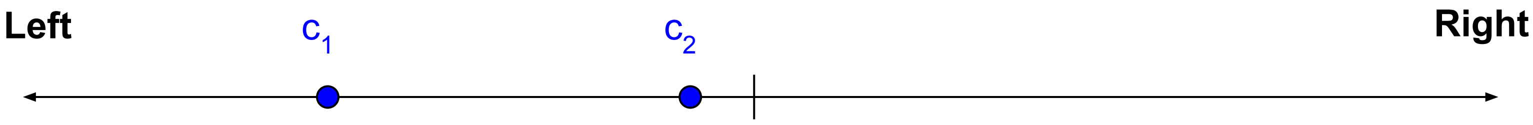

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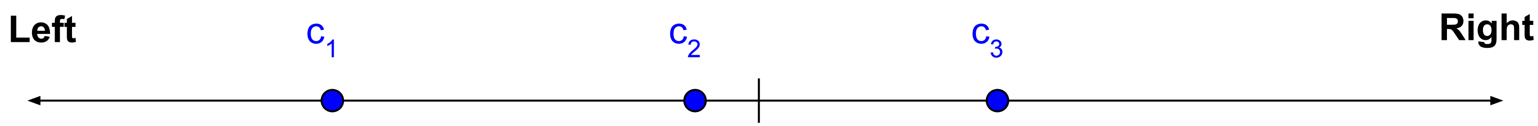

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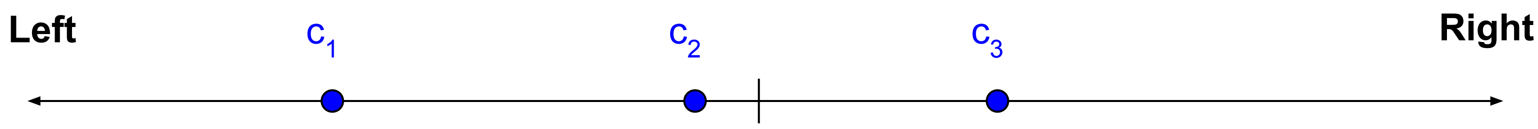

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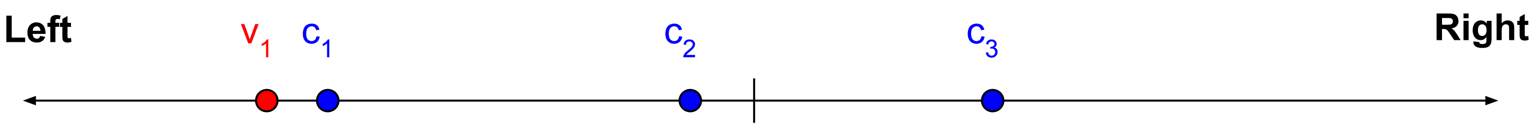

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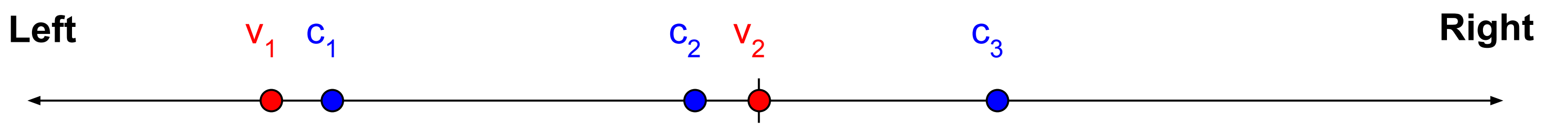

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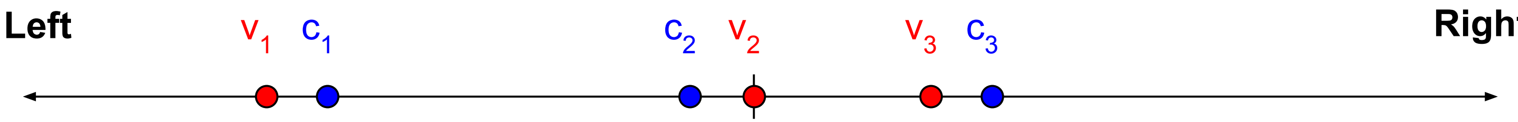

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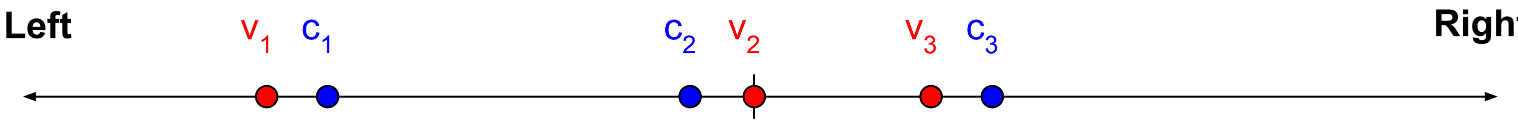

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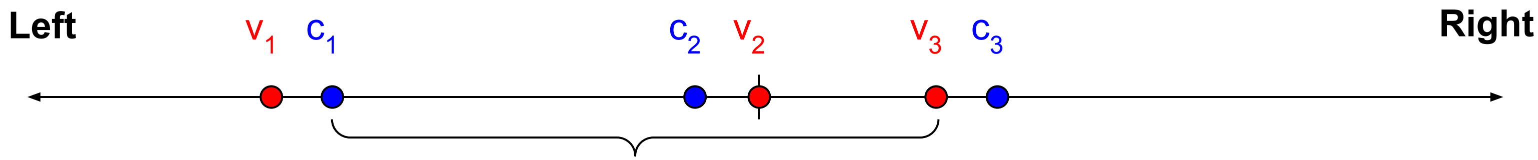

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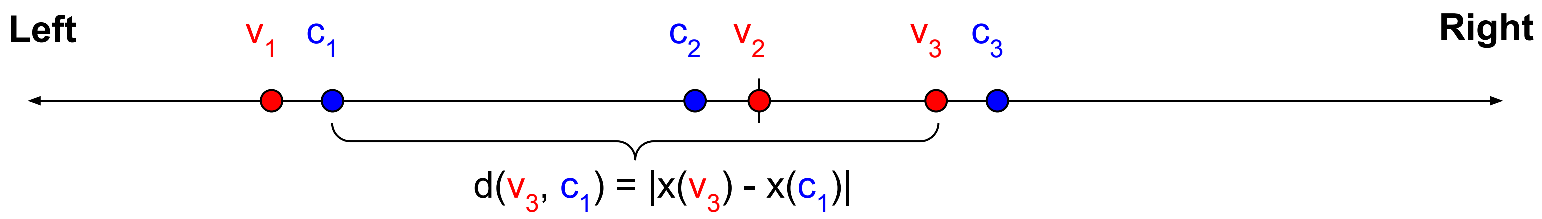

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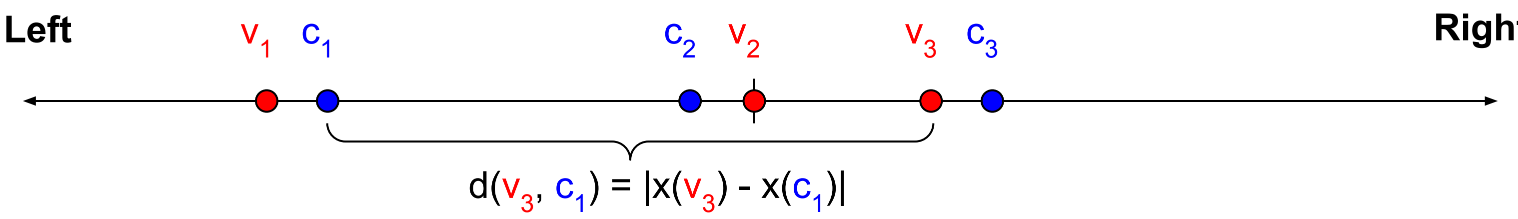

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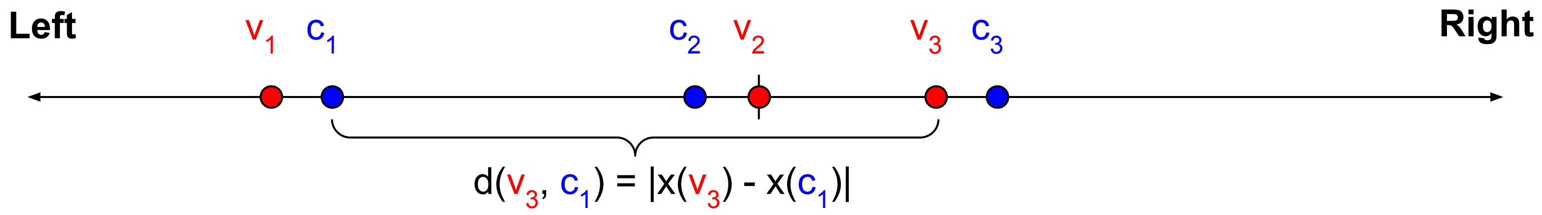

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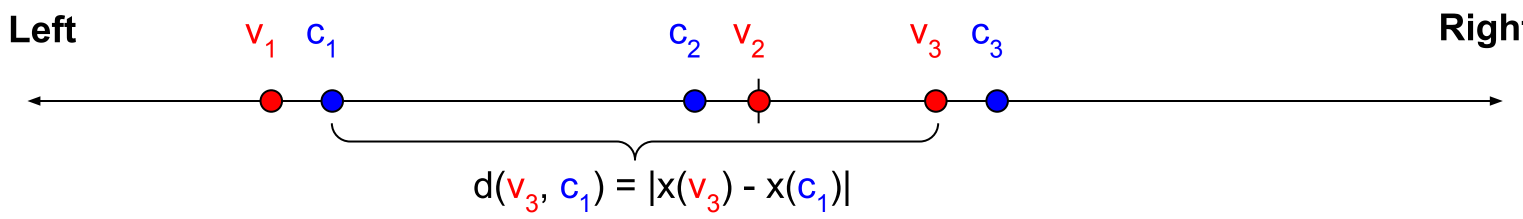

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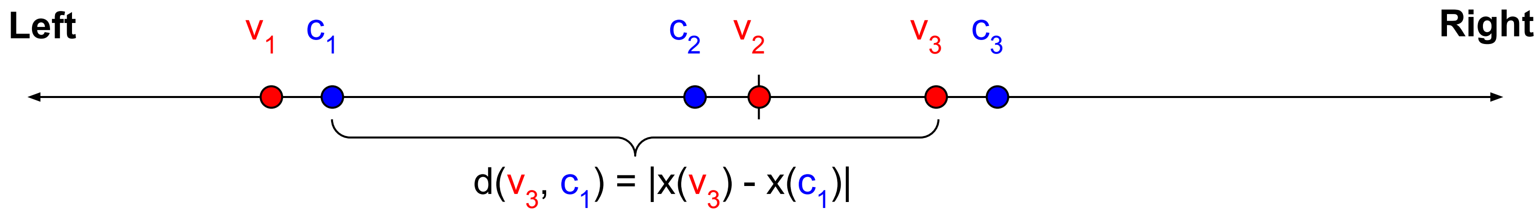

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Majority Tournament

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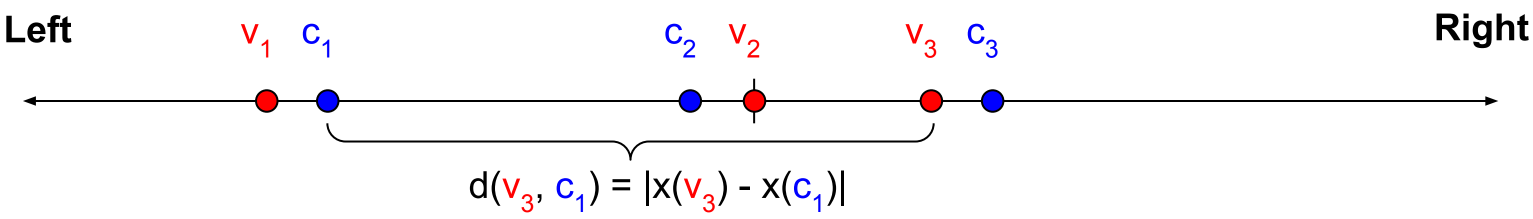

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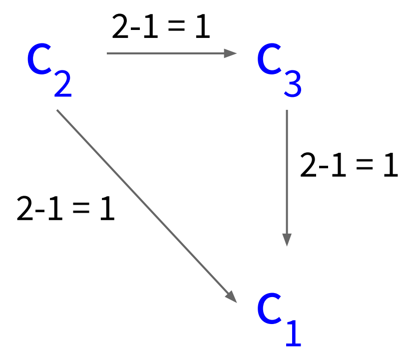

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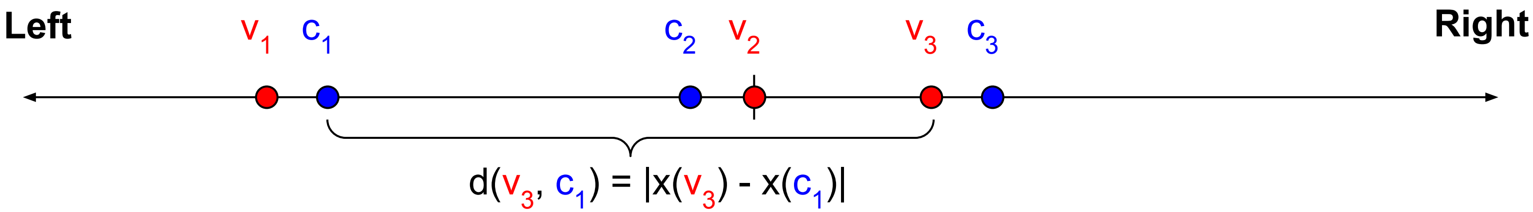

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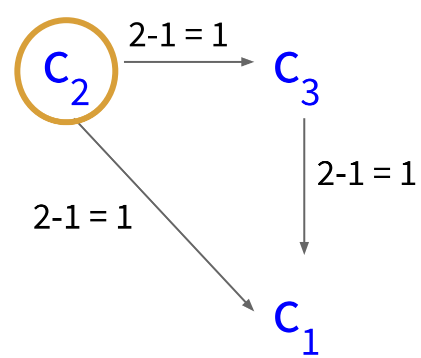

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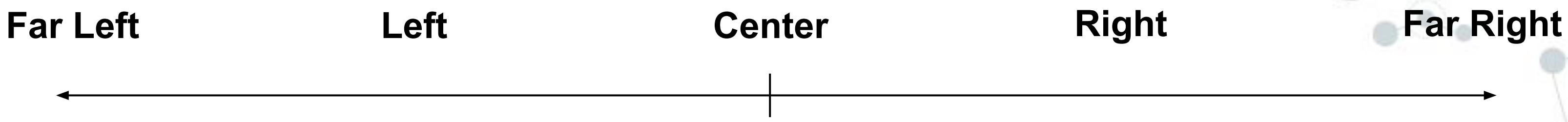

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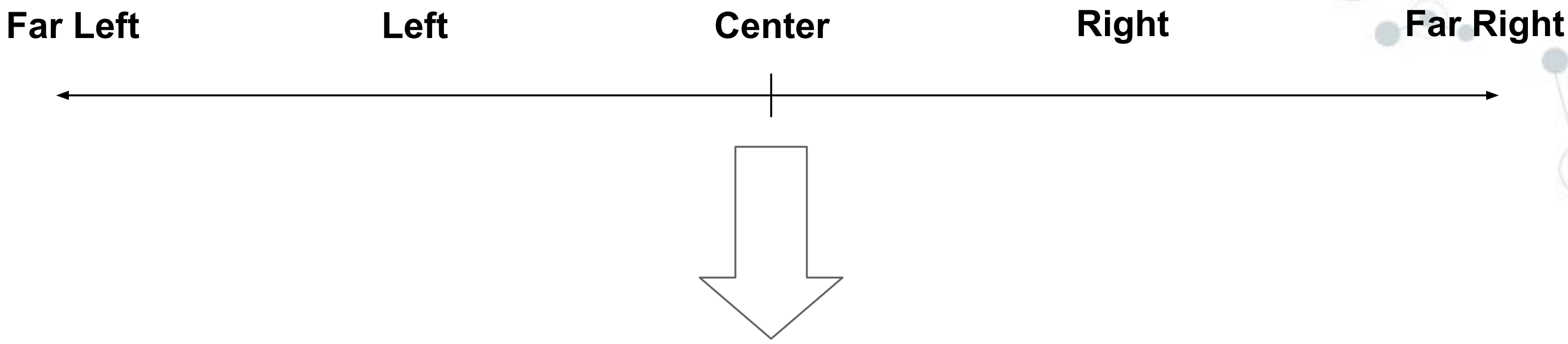

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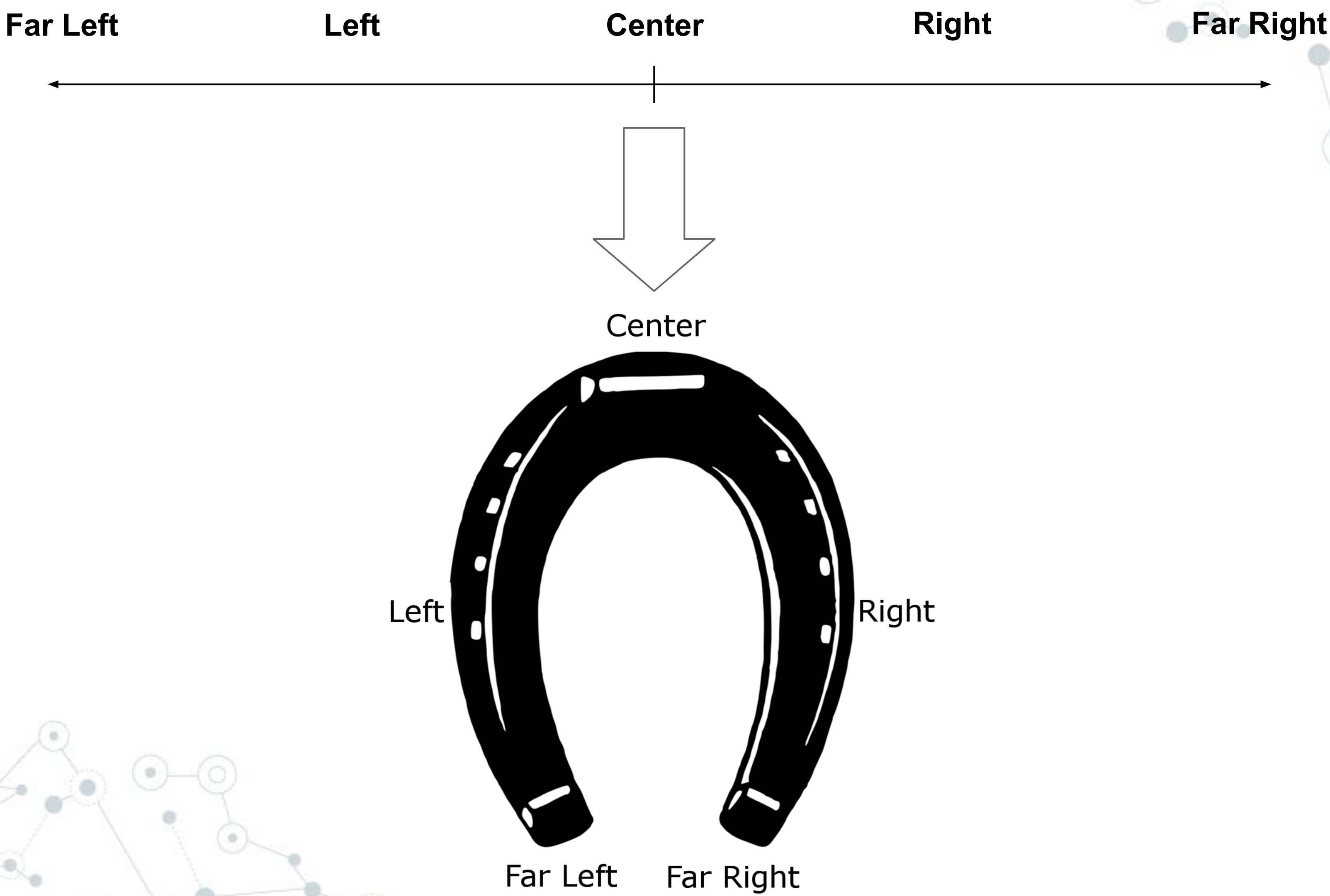

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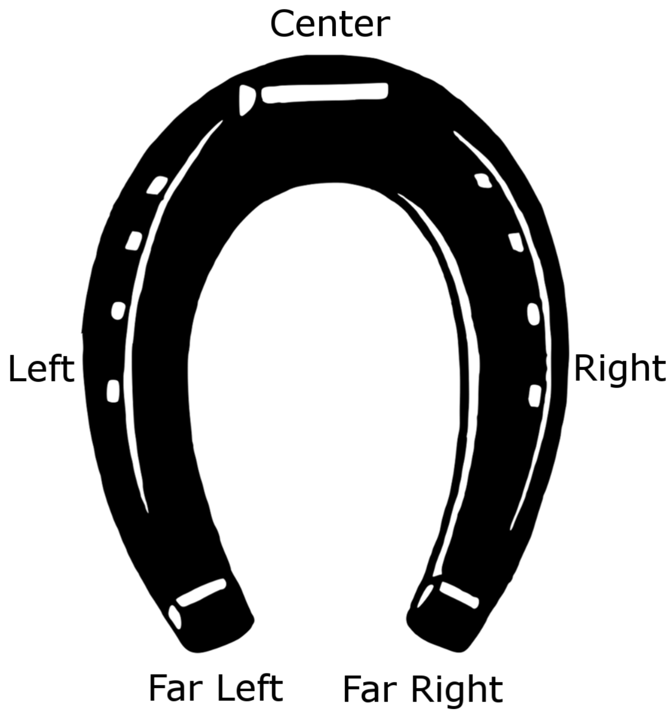

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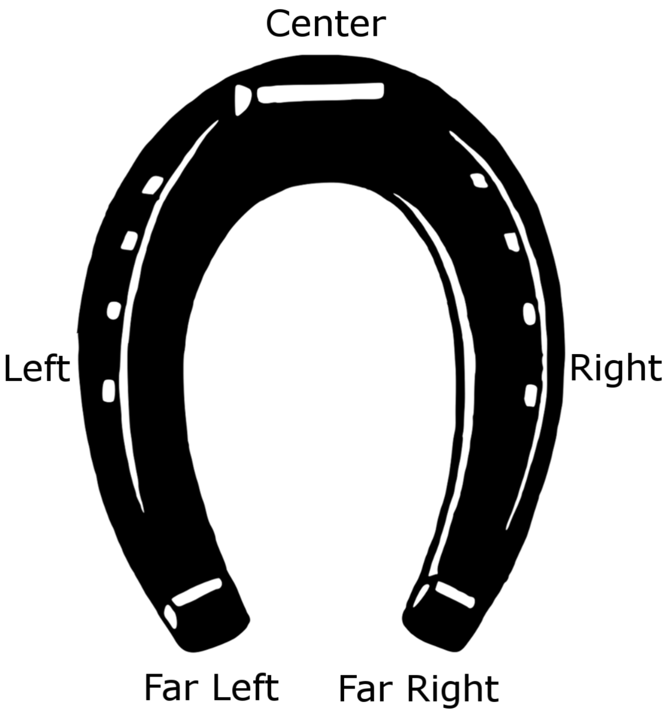

"Unholy Alliance"

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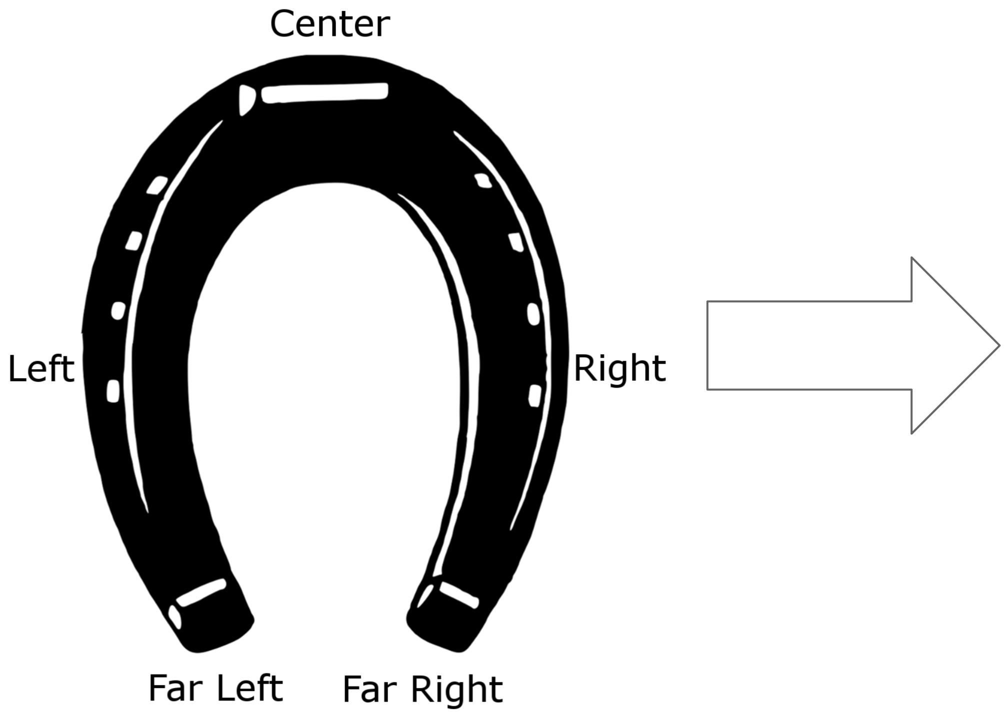

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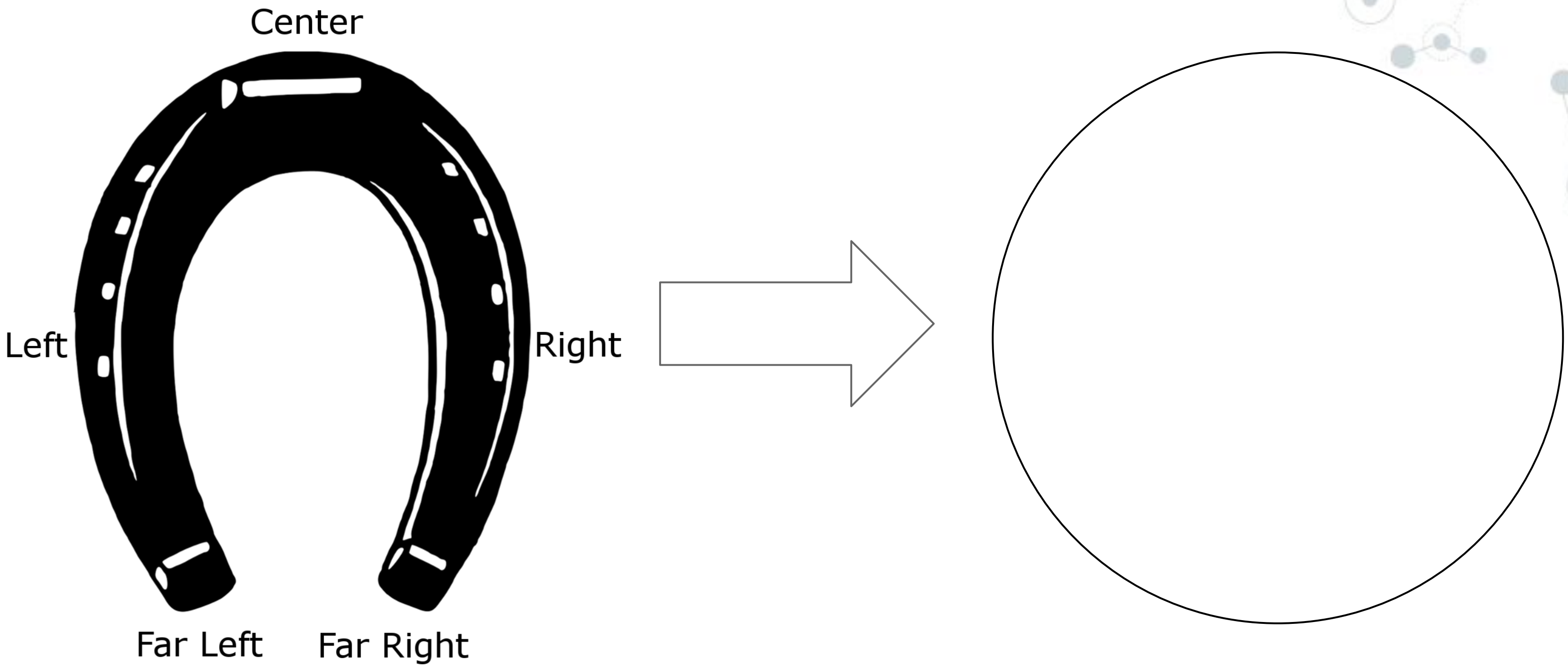

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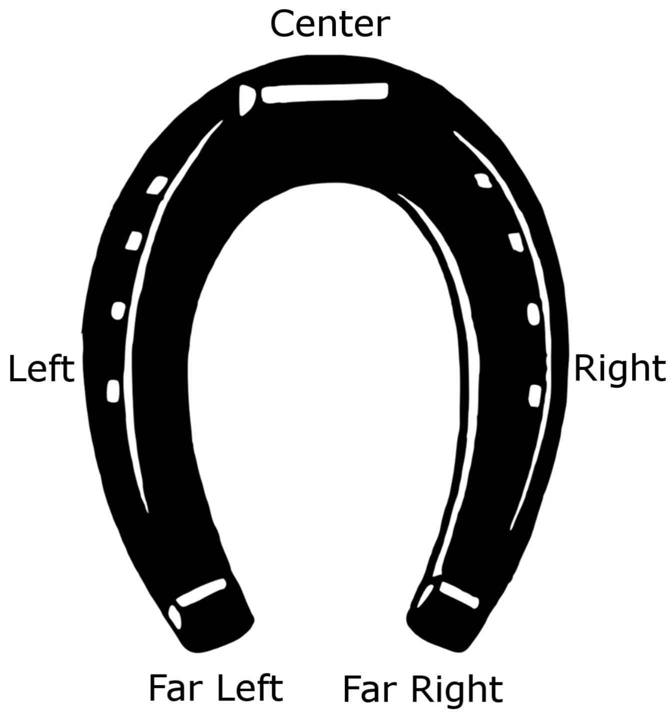

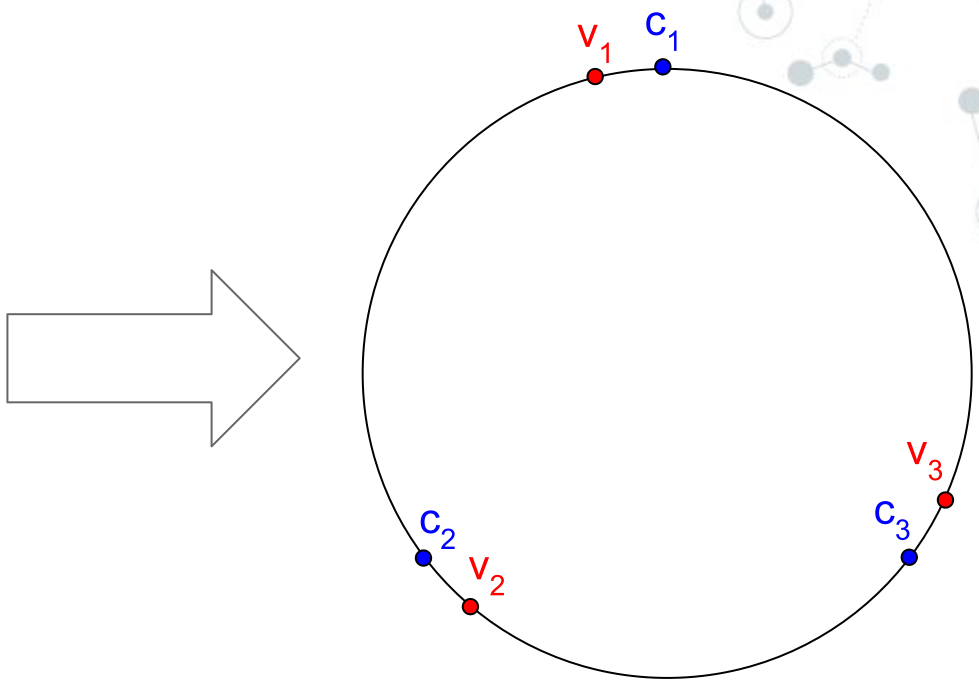

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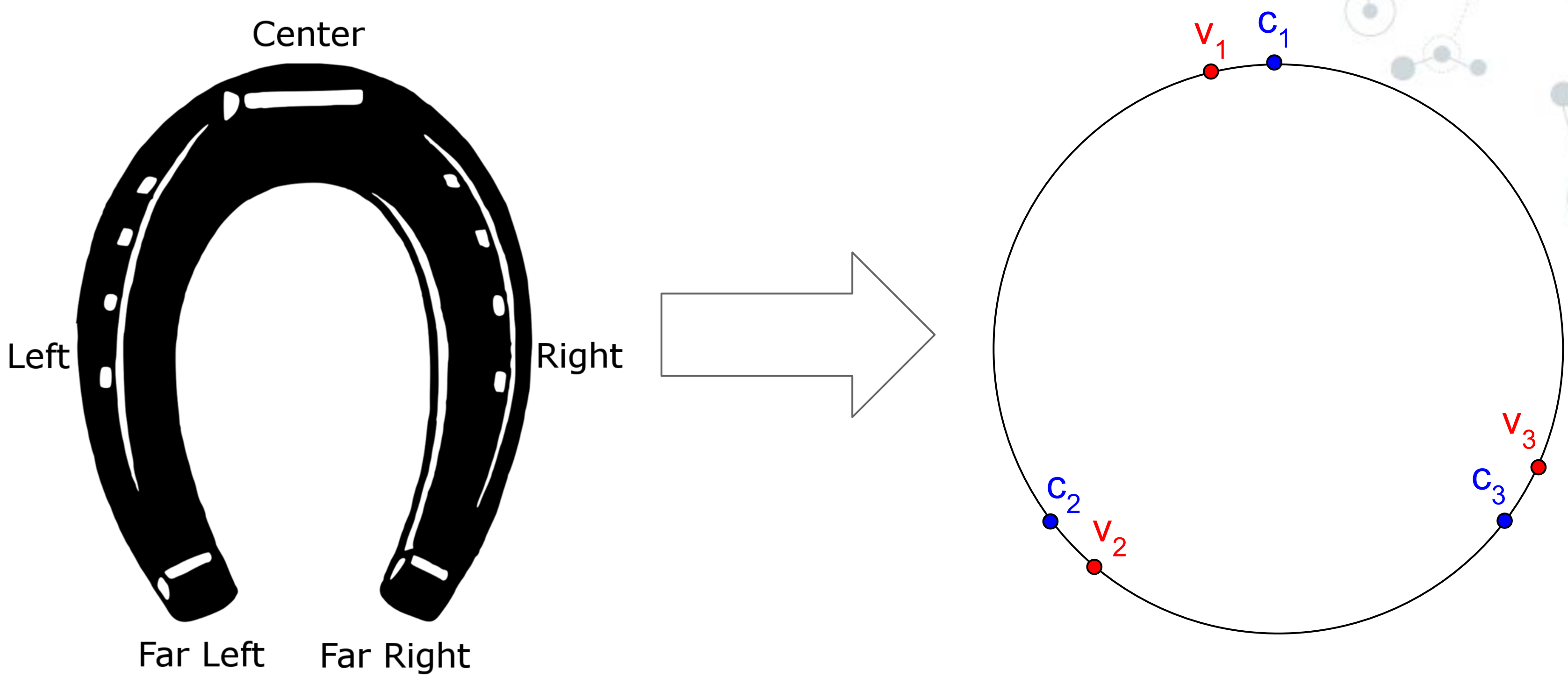

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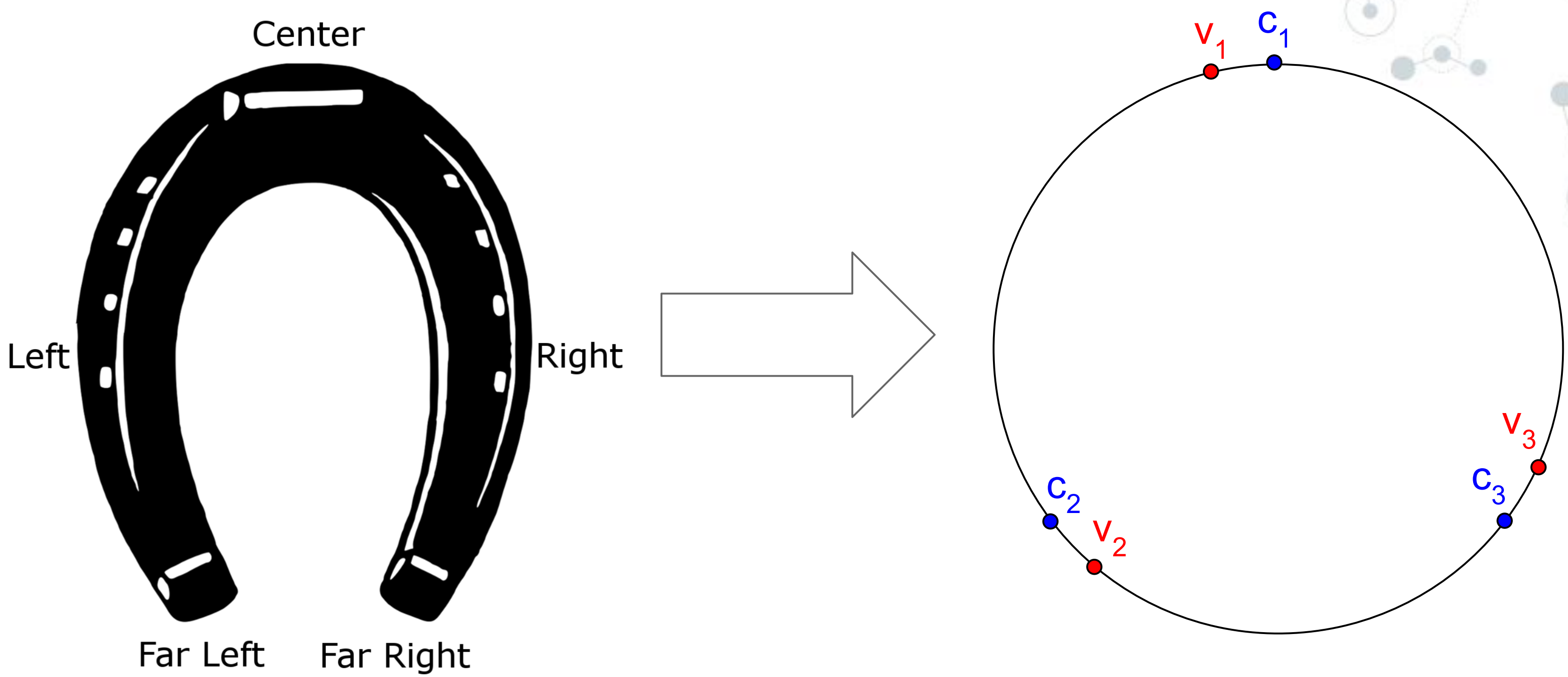

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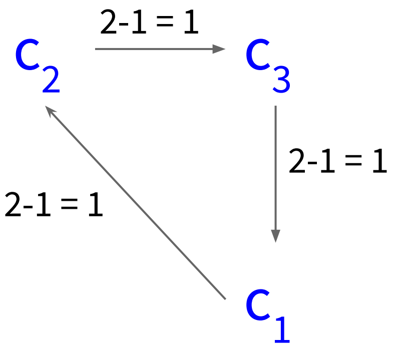

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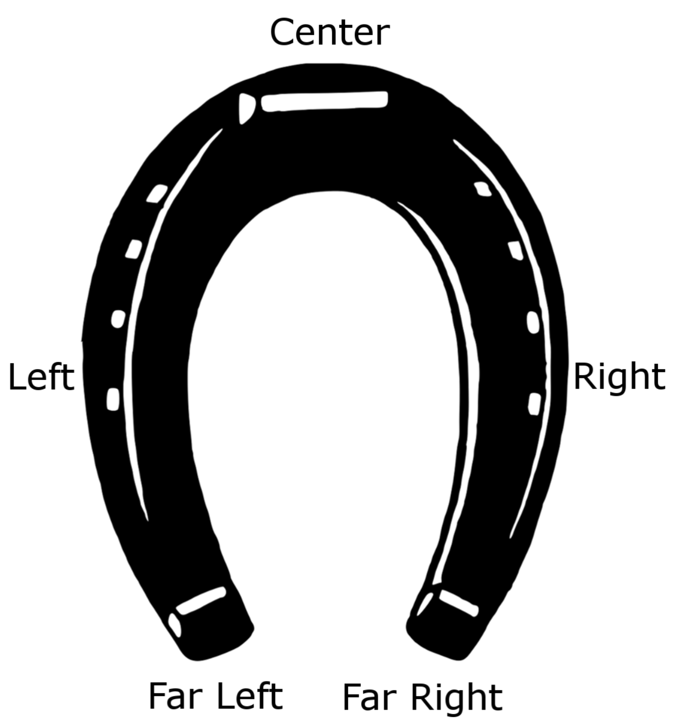

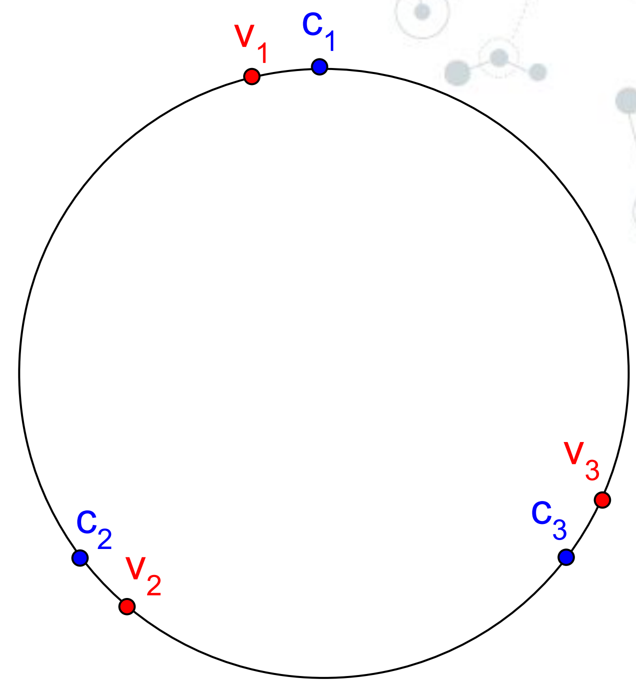

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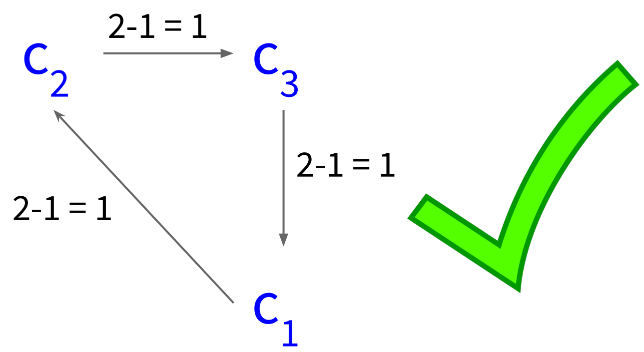

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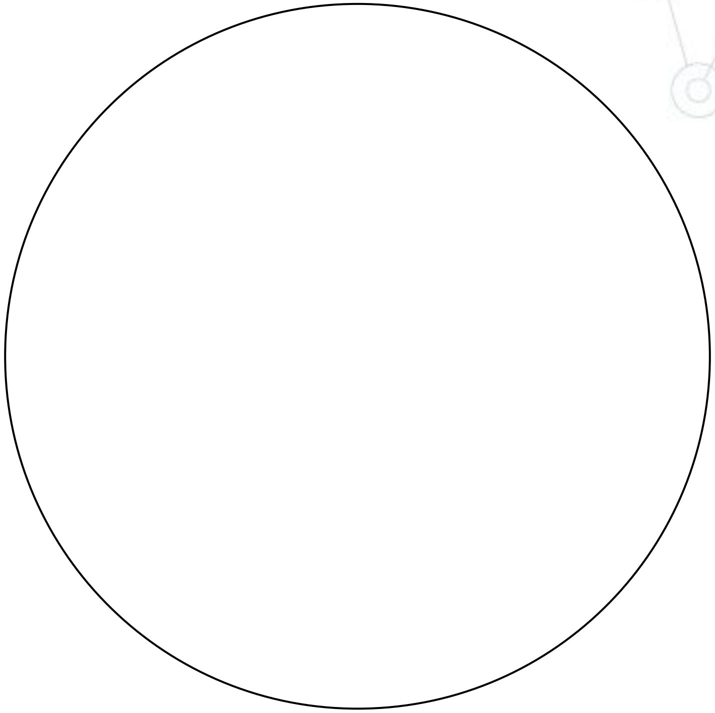

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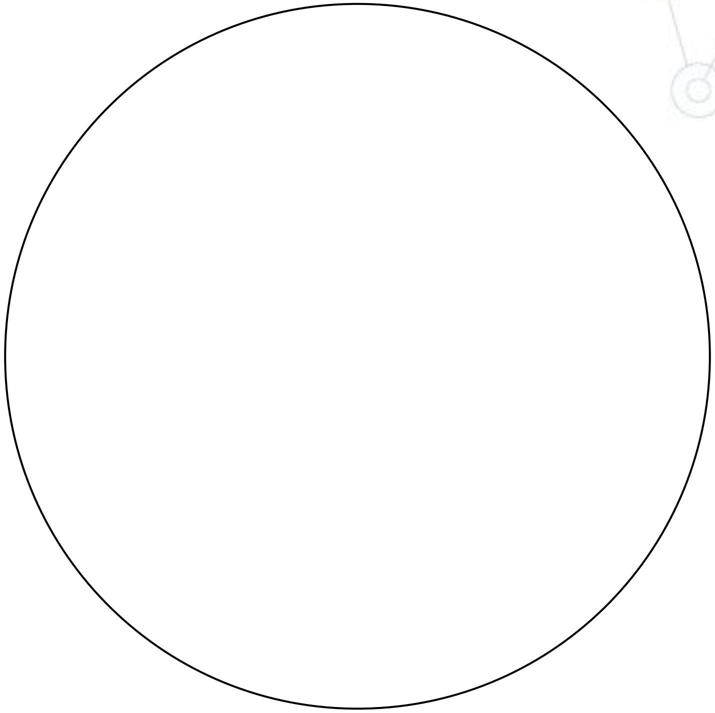

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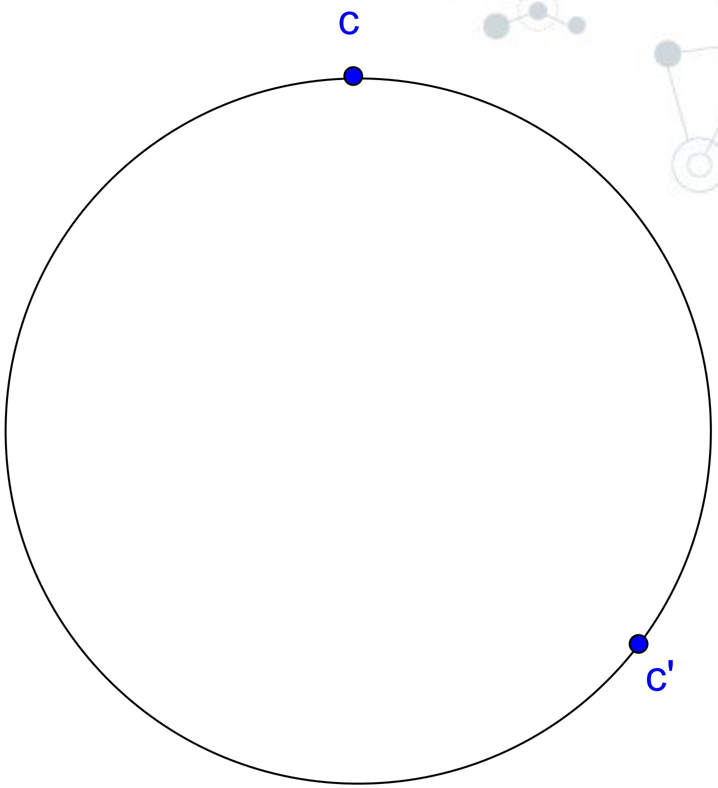

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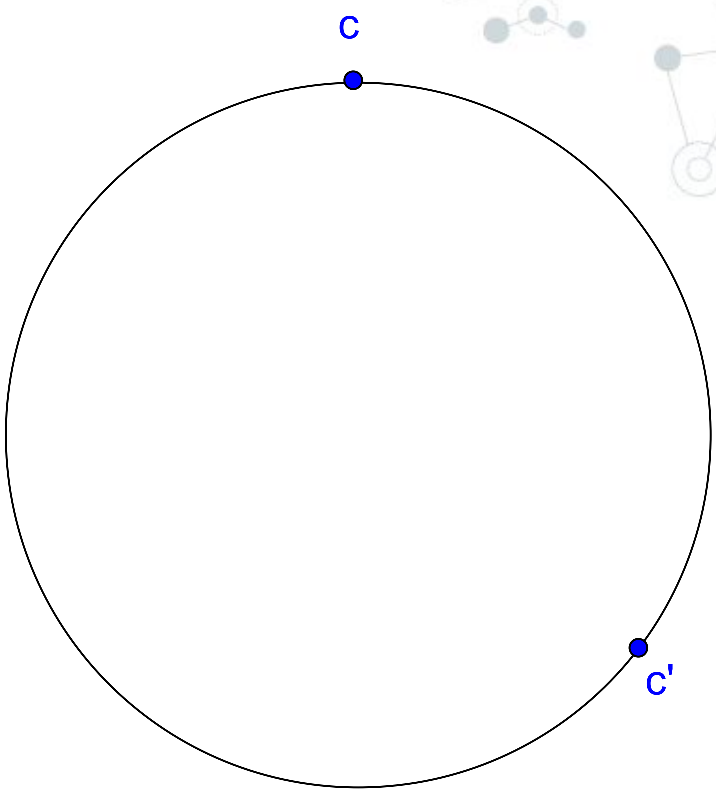

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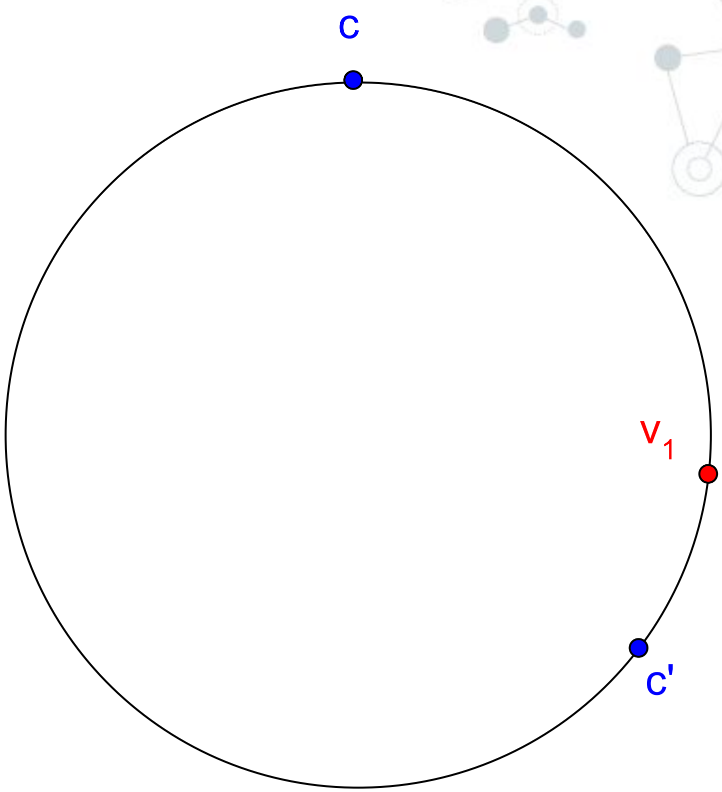

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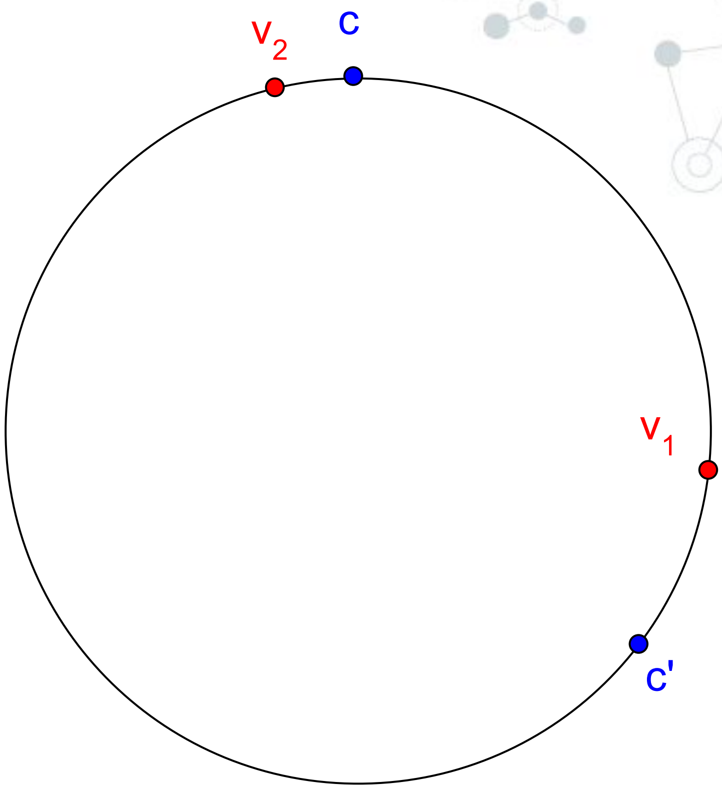

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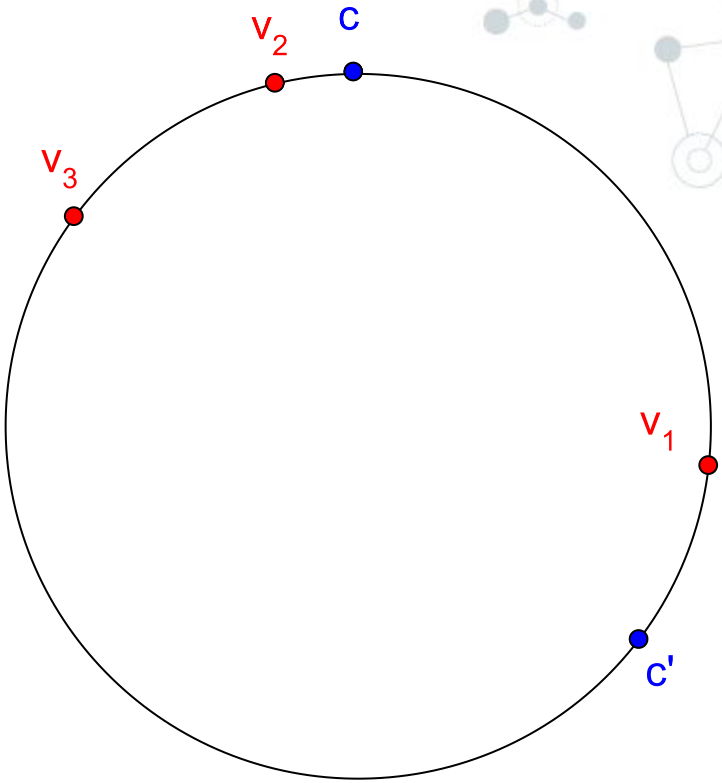

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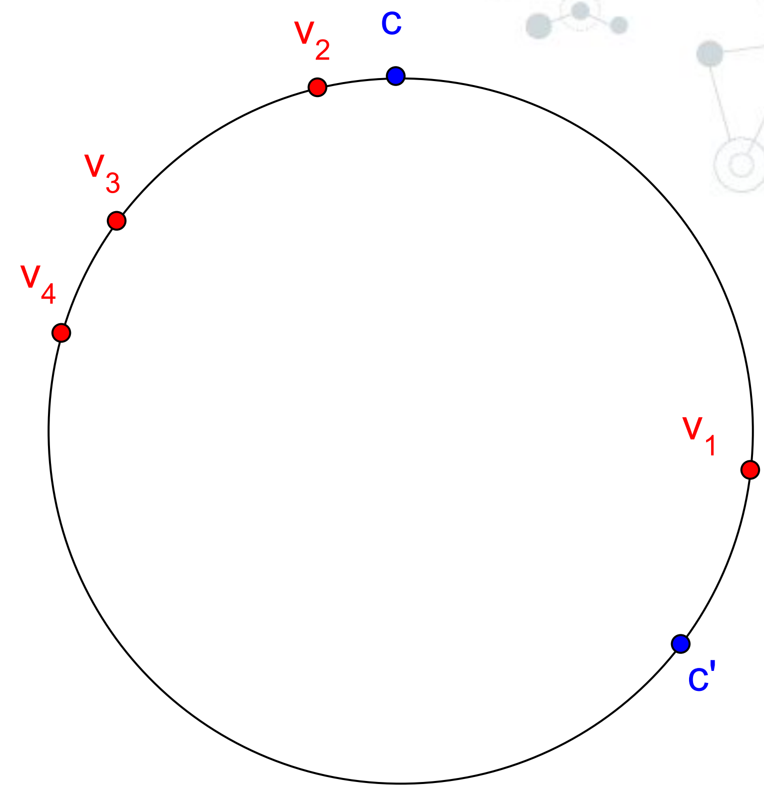

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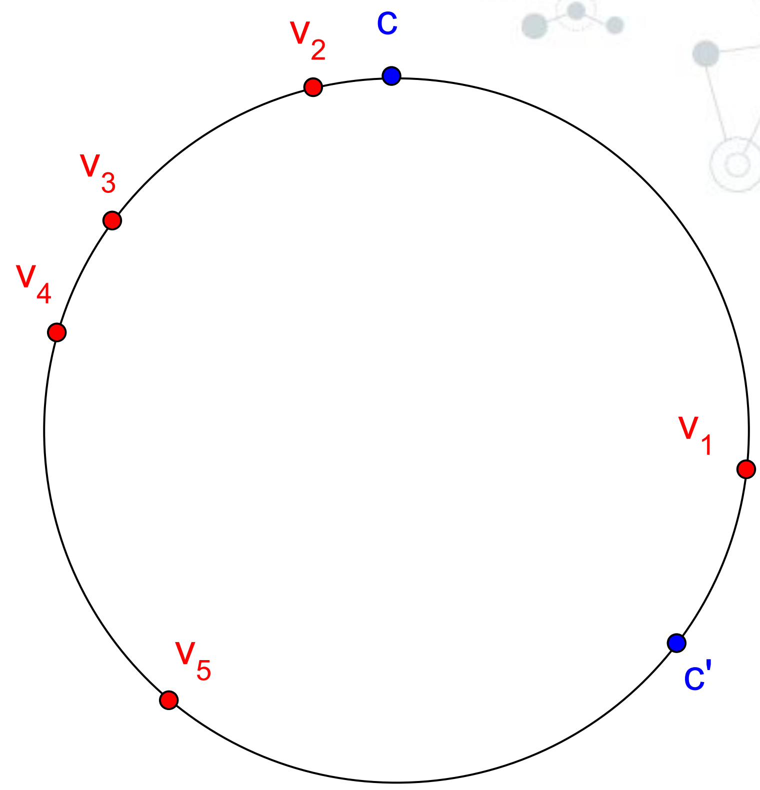

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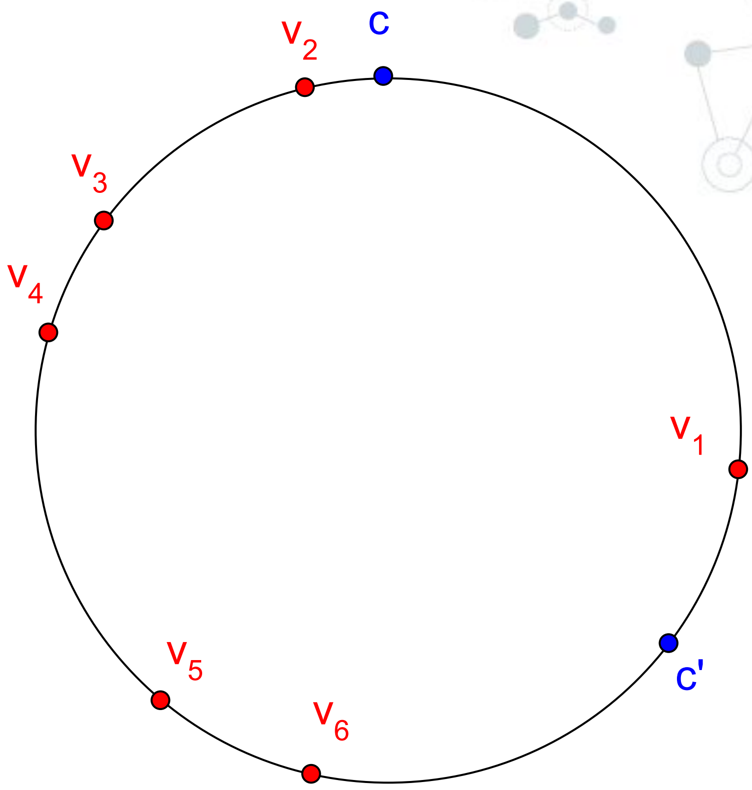

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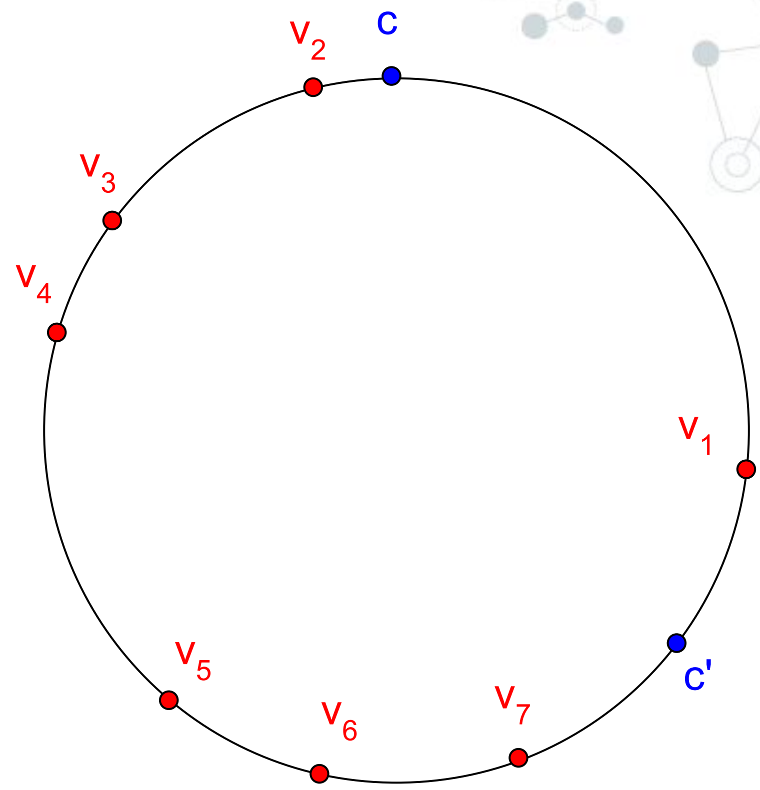

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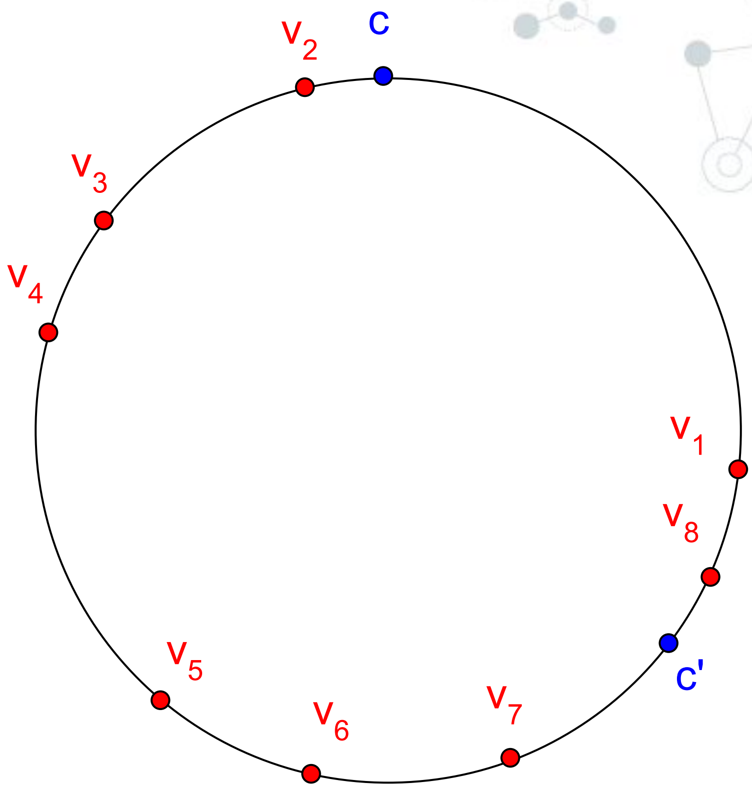

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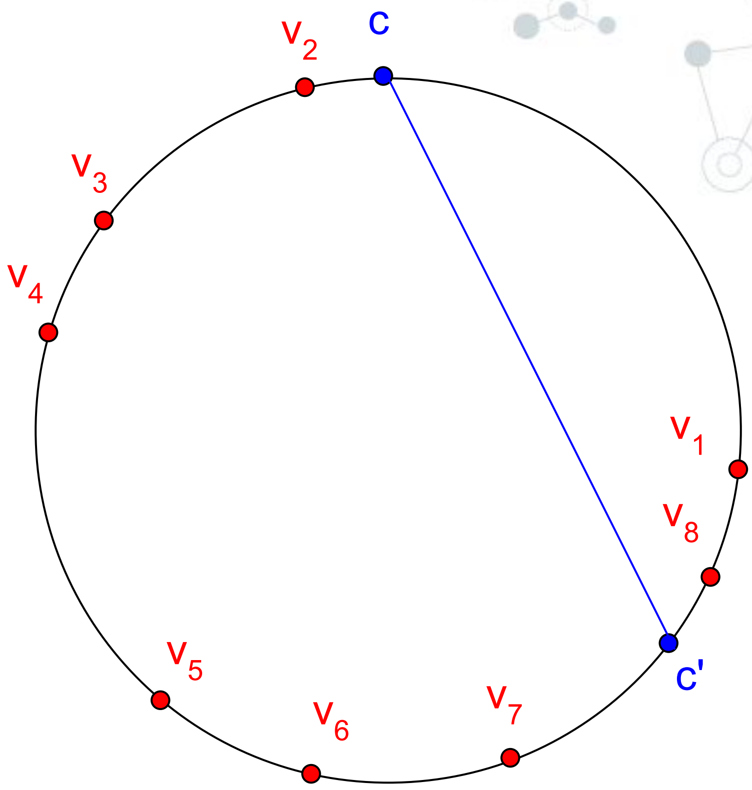

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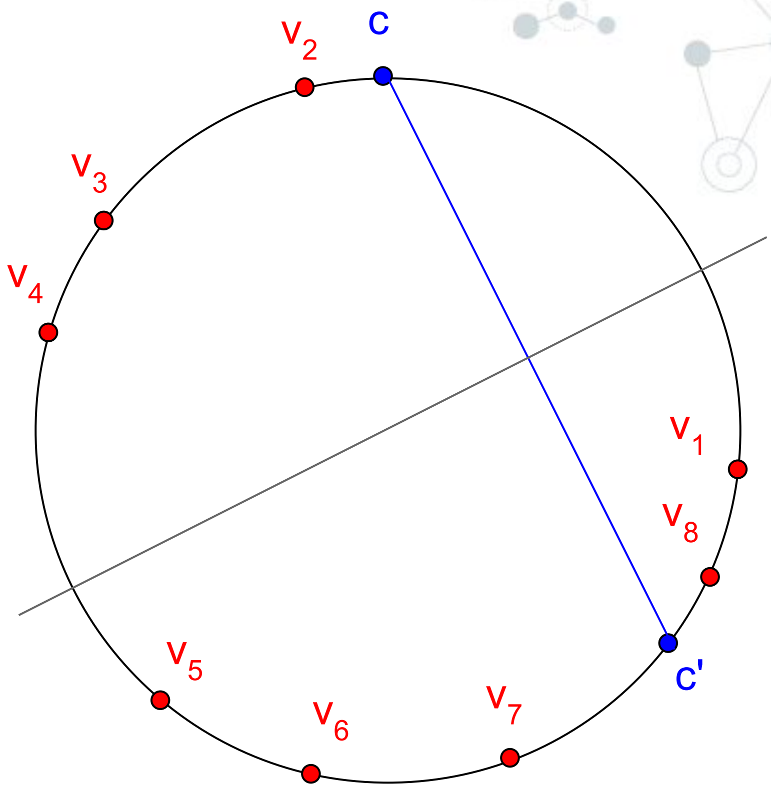

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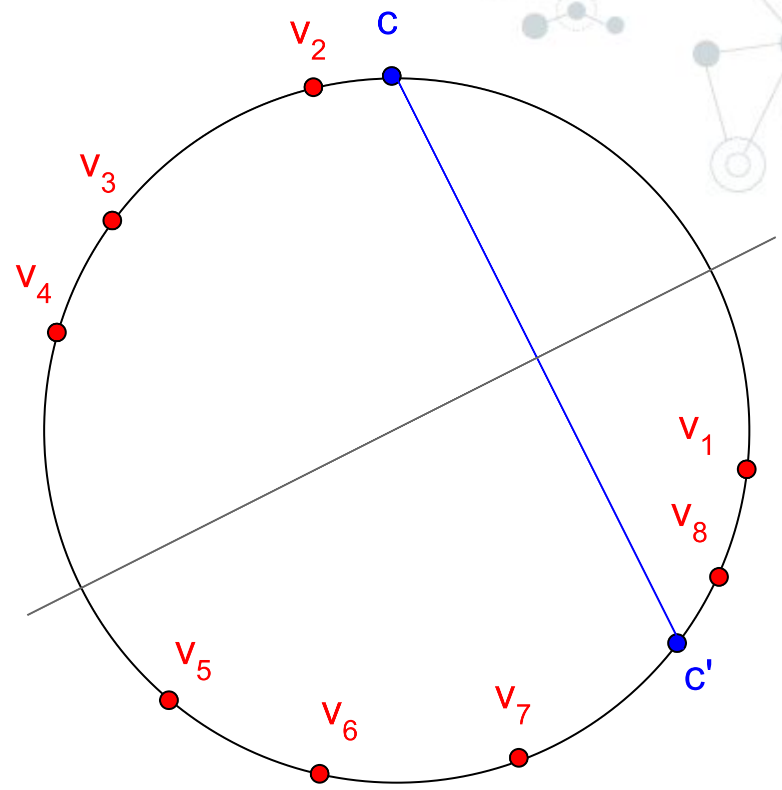
$c \succ c'$	0	1	1	1	0	0	0	0



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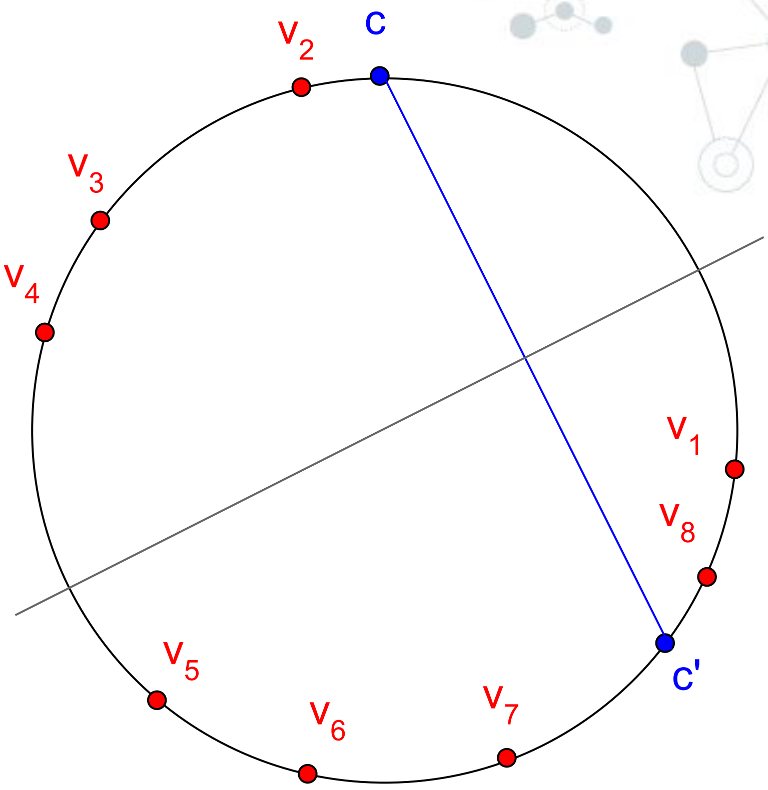
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
c ≻ c'	0	1	1	1	0	0	0	0



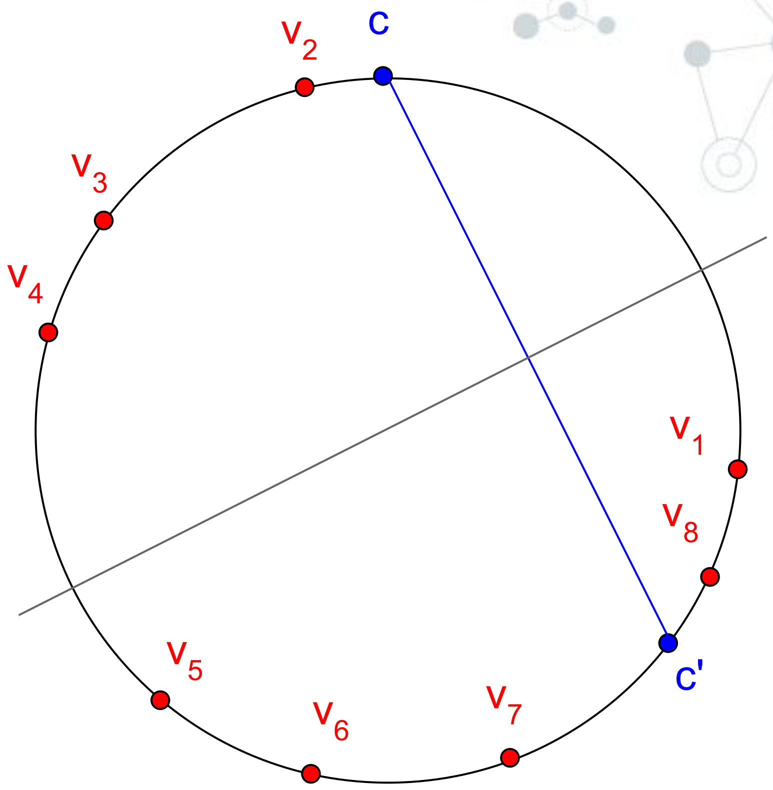
Voters preferring c to c'
form a "circular" interval!

A More General Property

Consider candidates c and c'.

And voters sorted by angle.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
c ≻ c'	0	1	1	1	0	0	0	0
c' ≻ c	1	0	0	0	1	1	1	1



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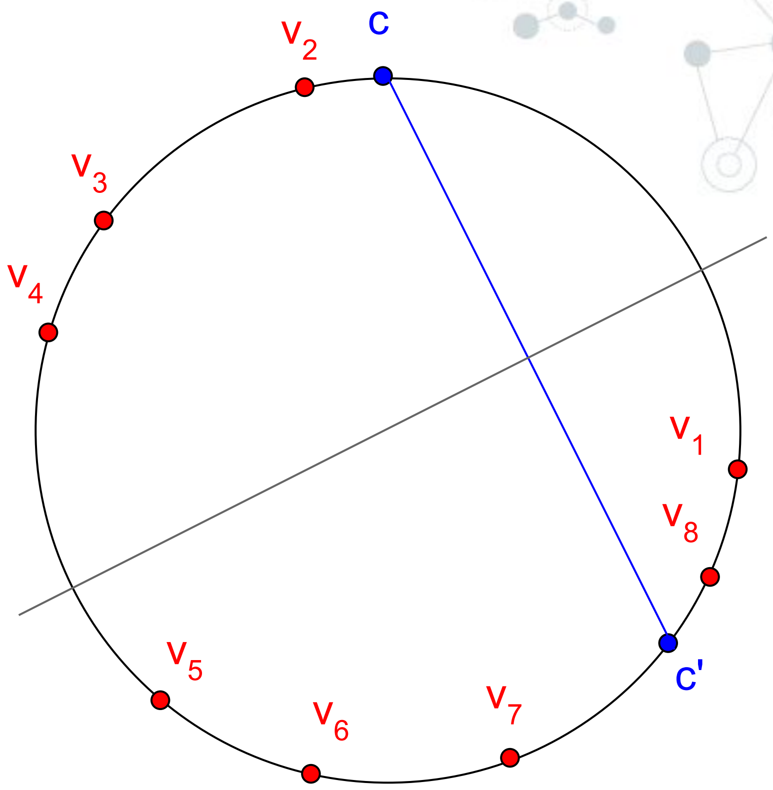

Voters preferring c to c' form a "circular" interval!

⇔ At most 2 switches per row.

k-Crossing Elections

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	v_1	v_2	v_3	v_4
$c_1 \succ c_2$	1	0	0	1
$c_2 \succ c_3$	1	0	1	0
$c_1 \succ c_3$	1	0	0	0

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$c_2 \succ c_3$	1	0	1	0
$c_1 \succ c_3$	1	0	0	0

	v_1	v_3	v_2	v_4
$c_1 \succ c_2$	1	0	0	1
$c_2 \succ c_3$	1	1	0	0
$c_1 \succ c_3$	1	0	0	0

2.

Recognition

Using the Consecutive Ones Problem

Recognition

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Deciding whether an election is k-crossing.

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  [Elkind et al., 2012; Bredereck et al., 2013].
- Two-crossing: *poly-time*
  Reduction to consecutive ones **(this paper)**.
- k-crossing: *open*
  We conjecture NP-complete for $k \geq 4$.

Recognition for Two-Crossing

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Given candidates $c_1, \ldots, c_M$ and voters $v_1, \ldots, v_N$

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Then, check whether columns can be permuted s.t. 1s in each row form a continuous **circular** run.

	v_1	v_2	v_3	v_4
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$c_2 \succ c_3$	1	0	1	0
$c_1 \succ c_3$	1	0	0	0

$v_1 : c_1 \succ c_2 \succ c_3$

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$c_1 \succ c_2$	1	**0**	**0**	1
$c_2 \succ c_3$	1	**1**	**0**	0
$c_1 \succ c_3$	1	**0**	**0**	0

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$v_4 : c_3 > c_1 > c_2$

$O(NM^2)$

	v_1	v_2	v_3	v_4
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$c_1 > c_3$	1	0	0	0

	v_1	v_3	v_2	v_4
$c_1 > c_2$	1	**0**	**0**	1
$c_2 > c_3$	1	**1**	**0**	0
$c_1 > c_3$	1	**0**	**0**	0

3.

Majority Tournament Universality

And NP-Hardness of Kemeny

Weighted Majority Tournament

Two-crossing: **also any (weighted) tournament can be obtained!**

Weighted Majority Tournament

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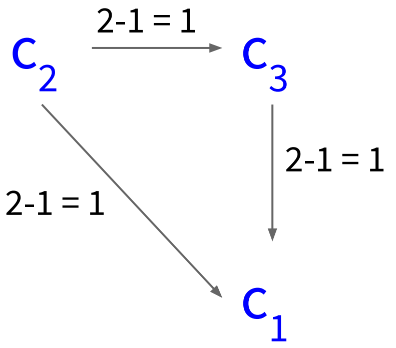
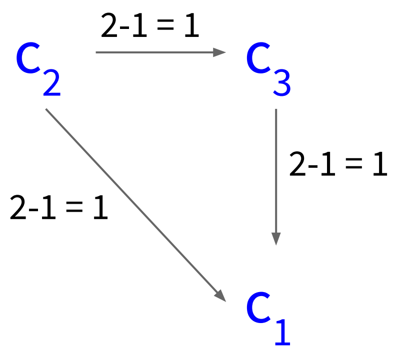


Weighted Majority Tournament

Single-crossing: tournament is transitive.

Two-crossing: **also any (weighted) tournament can be obtained!**



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General elections: any (weighted) tournament can be obtained.

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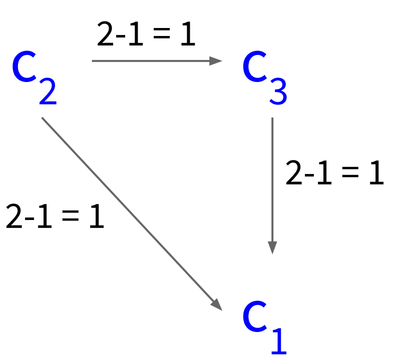

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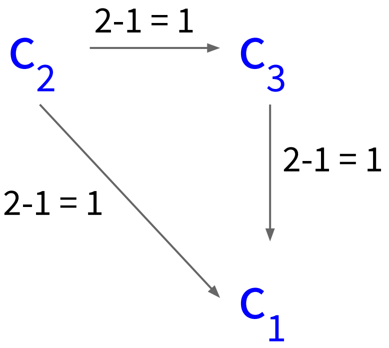

Proof

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Construct the **“Double-BubbleSort”** profile.

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e.g. M = 4 candidates.

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v_1
1
2
3
4

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e.g. M = 4 candidates.

v_1	v_2
1	2
2	1
3	3
4	4

Proof

Construct the **“Double-BubbleSort”** profile.

e.g. M = 4 candidates.

v_1	v_2	v_3
1	2	2
2	1	3
3	3	1
4	4	4

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1	2	2	2
2	1	3	3
3	3	1	4
4	4	4	1

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v_1	v_2	v_3	v_4	v_5
1	2	2	2	3
2	1	3	3	2
3	3	1	4	4
4	4	4	1	1

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e.g. M = 4 candidates.

v_1	v_2	v_3	v_4	v_5	v_6
1	2	2	2	3	3
2	1	3	3	2	4
3	3	1	4	4	2
4	4	4	1	1	1

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v_1	v_2	v_3	v_4	v_5	v_6	v_7
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2	1	3	3	2	4	3
3	3	1	4	4	2	2
4	4	4	1	1	1	1

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1	2	2	2	3	3	4	4	4
2	1	3	3	2	4	3	3	1
3	3	1	4	4	2	2	1	3
4	4	4	1	1	1	1	2	2

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1	2	2	2	3	3	4	4	4	1
2	1	3	3	2	4	3	3	1	4
3	3	1	4	4	2	2	1	3	3
4	4	4	1	1	1	1	2	2	2

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v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
1	2	2	2	3	3	4	4	4	1	1
2	1	3	3	2	4	3	3	1	4	4
3	3	1	4	4	2	2	1	3	3	2
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e.g. M = 4 candidates.

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1	2	2	2	3	3	4	4	4	1	1	1
2	1	3	3	2	4	3	3	1	4	4	2
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This profile is two-crossing!

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$1 \xrightarrow{1} 3$

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Proof

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e.g. M = 4 candidates.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
1	2	2	2	3	3	4	4	4	1	1	1	1
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2	1	3	3	2	4	3	3	1	4	4	2	2
3	3	1	4	4	2	2	1	3	3	2	4	3
4	4	4	1	1	1	1	2	2	2	3	3	4

V

$1 \xrightarrow{1} 3$

This profile is two-crossing!

Proof

Construct the **“Double-BubbleSort”** profile.

e.g. M = 4 candidates.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
1	2	2	2	3	3	4	4	4	1	1	1	1
2	1	3	3	2	4	3	3	1	4	4	2	2
3	3	1	4	4	2	2	1	3	3	2	4	3
4	4	4	1	1	1	1	2	2	2	3	3	4

V

V'

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v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
1	2	2	2	3	3	4	4	4	1	1	1	1
2	1	3	3	2	4	3	3	1	4	4	2	2
3	3	1	4	4	2	2	1	3	3	2	4	3
4	4	4	1	1	1	1	2	2	2	3	3	4

v

v'

$1 \xrightarrow{1} 3$

This profile is two-crossing!

Proof

Construct the **"Double-BubbleSort"** profile.

e.g. M = 4 candidates.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
1	2	2	2	3	3	4	4	4	1	1	1	1
2	1	3	3	2	4	3	3	1	4	4	2	2
3	3	1	4	4	2	2	1	3	3	2	4	3
4	4	4	1	1	1	1	2	2	2	3	3	4

V

V'

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This profile is two-crossing!

Consequences: NP-hardness

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Thus, NP-hardness results carry over to two-crossing:

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- **Kemeny and Slater are NP-hard.**

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Thus, NP-hardness results carry over to two-crossing:

- **Kemeny and Slater are NP-hard.**
- Banks, Minimal Extending Set, Tournament Equilibrium Set and Ranked Pairs also NP-hard.

4.

Young's Rule

Using Total Unimodularity

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- NP-hard in general:
  [Rothe et al., 2003; Brandt et al., 2015; Fitzsimmons and Hemaspaandra, 2020].

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- NP-hard in general: [Rothe et al., 2003; Brandt et al., 2015; Fitzsimmons and Hemaspaandra, 2020].
- Two-crossing: **scores in poly-time (this paper)**.

Young's Rule

The natural LP does not have integer vertices.

By fixing the number of voters to keep we arrive at an LP with integer vertices, so we can solve the LP.

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By reducing to negative weight cycle detection we further **improve the running time to $O((n + m^2)n^{3/2} \log n)$**.

5.

Chamberlin-Courant Rule

Using Dynamic Programming

Representation

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In an election we need to select a committee of K candidates to best represent the electorate.

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In an election we need to select a committee of K candidates to best represent the electorate.

v_1 : Blue > Yellow > Red > Pink > Green

v_2 : Yellow > Green > Red > Pink > Blue

v_3 : Green > Red > Blue > Pink > Yellow

Representation

In an election we need to select a committee of K candidates to best represent the electorate.

e.g. **K = 2**

v_1 : Blue > Yellow > Red > Pink > Green

v_2 : Yellow > Green > Red > Pink > Blue

v_3 : Green > Red > Blue > Pink > Yellow

Representation

In an election we need to select a committee of K candidates to best represent the electorate.

e.g. **K = 2**

v_1	:		>	Yellow	>		>	Pink	>	
v_2	:	Yellow	>		>		>	Pink	>	
v_3	:		>		>		>	Pink	>	Yellow

Representation

In an election we need to select a committee of K candidates to best **represent** the electorate.

e.g. **K = 2**

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Representation

In an election we need to select a committee of K candidates to best **represent** the electorate.

e.g. **K = 2**

v_1 : > Yellow > > Pink >

v_2 : Yellow > > > Pink >

v_3 : > > > Pink > Yellow

Q: How to compare K-committees?

The Chamberlin-Courant Rule

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Voters specify their *dissatisfaction* with each candidate.

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	0	1	5	8	9
v_1 :	Blue >	Yellow >	Red >	Pink >	Green
	0	3	3	4	8
v_2 :	Yellow >	Green >	Red >	Pink >	Blue
	0	1	1	2	3
v_3 :	Green >	Red >	Blue >	Pink >	Yellow

The Chamberlin-Courant Rule

Voters specify their *dissatisfaction* with each candidate.

Pick the K-committee that **minimizes** the total/maximum dissatisfaction.

		0		1		5		8		9
v_1	:	Blue	>	Yellow	>	Red	>	Pink	>	Green
		0		3		3		4		8
v_2	:	Yellow	>	Green	>	Red	>	Pink	>	Blue
		0		1		1		2		3
v_3	:	Green	>	Red	>	Blue	>	Pink	>	Yellow

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				1				8		3	
v_1	:		>	Yellow	>		>	Pink	>		
		0						4			
v_2	:	Yellow	>		>		>	Pink	>		
								2		3	
v_3	:		>		>		>	Pink	>	Yellow	

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v_2 :	$\boxed{\overset{0}{\text{Yellow}}}$	>		>		>	$\overset{4}{\text{Pink}}$	>	
v_3 :		>		>		>	$\boxed{\overset{2}{\text{Pink}}}$	>	$\overset{3}{\text{Yellow}}$

Total = **3** (Utilitarian-CC) **- in this talk.**

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v_1	:		>	Yellow	>		>	Pink	>	
		0						4		
v_2	:	Yellow	>		>		>	Pink	>	
								2		3
v_3	:		>		>		>	Pink	>	Yellow

Total = **3** (Utilitarian-CC) **- in this talk.**

Maximum = **2** (Egalitarian-CC) [Betzler et al.; 2013]

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Both polynomial for two-crossing (this paper).

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v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
r(v)	B	R	R	Y	R	P	P	G

Decomposition For Two-Crossing

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v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
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R splits

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G

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B	O	B

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P	P

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R splits

v_1		v_3	v_4	v_5		v_7	v_8		v_{10}
G		B	O	B		P	P		Y

B splits

Decomposition For Two-Crossing

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
G	R	B	O	B	R	P	P	R	Y

R splits

v_1
G

v_3	v_4	v_5
B	O	B

v_7	v_8
P	P

v_{10}
Y

B splits

v_4
O

Decomposition For Two-Crossing

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
G	R	B	O	B	R	P	P	R	Y

R splits

v_1
G

v_3	v_4	v_5
B	O	B

v_7	v_8
P	P

v_{10}
Y

B splits

v_4
O

There exists a decomposable optimal committee!

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5. Three-crossing and above in general?

Hope you enjoyed!